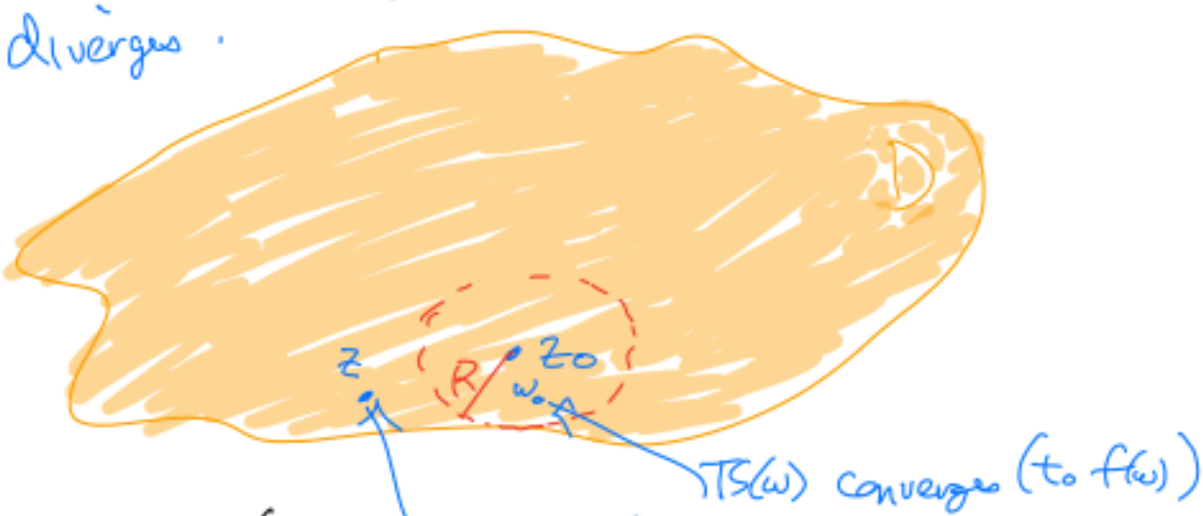
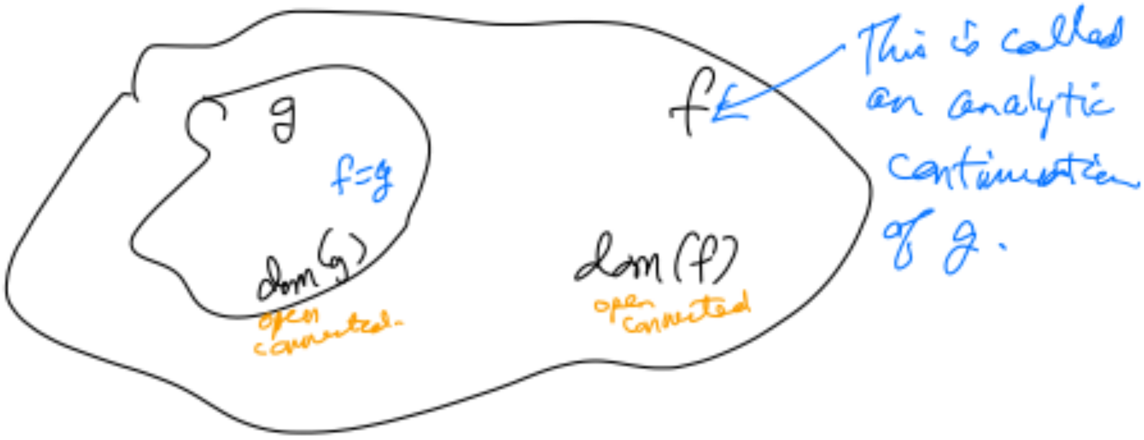
More comments about Laurent series &
Taylor series

(1) Suppose f is holomorphic on a domain $D \subseteq \mathbb{C}$, and $z_0 \in D$.
Then the Taylor series $TS(z) = \sum_{k=0}^{\infty} c_k (z-z_0)^k$ converges to f on the Disk $B(z_0, R)$, where R is the largest radius s.t. f's domain cannot be extended to contain any larger disk. (R can be ∞)
If R is finite, then if $|z-z_0| > R$, then $TS(z)$ diverges.



Note: $\overset{g(z) :=}{TS(z)}$ is a holomorphic function, just considered as the value of the series.
$f(z)$ agrees with $g(z)$ on $B(z_0, R)$ but has a larger domain.

Whenever this happens, we say that $f(z)$ is an analytic continuation of $g(z)$



Note: (Uniqueness of analytic continuation)

Since every pt of dom(g) is a limit pt, by the uniqueness theorem, f is the only possible analytic continuation of g to the domain dom(f).

Similarly, if $g(z)$ = Laurent Series of f on some annulus, again f is an analytic continuation of g (and is the only possible one on f's domain.)

$\boxed{\text{Ex}}$ (Have to be careful).

Let $f(z) = \mathrm{Log}(z)$ on $D = \{z \in \mathbb{C} \setminus \{0\} \mid 0 < \arg(z) < \frac{\pi}{2}\}$ (first quadrant).

Let $g(z) = \log(z)$ on $E = \{z \in \mathbb{C} \setminus \{0\} \mid 0 < \arg(z) < 2\pi\}$

argument chosen so $0 < \arg(z) < 2\pi$.

Let $h(z) = \underline{\mathrm{Log}(z)}$ on $F = \{z \in \mathbb{C} \setminus \{0\} \mid -\pi < \arg(z) < \pi\}$

$\arg(z)$ chosen s.t. $-\pi < \arg(z) < \pi$

Observe: both of these fcns are analytic continuations of f. However, $g(-i) = i\frac{3\pi}{2}$

and $h(-i) = -\frac{i\pi}{2}$.

Not a problem: dom g & dom h are distinct, and g is not defined on all of domain (h), and h is not defined on all of domain (g).

Lemma: Suppose $f(z) = \sum_{n \geq 0} a_n (z-z_0)^n$, which converges on $B(z_0, R)$. If $0 < r < R$, then $\sum_{n \geq 0} a_n (z-z_0)^n$ converges absolutely & uniformly on $B(z_0, r)$ (and $\overline{B(z_0,r)}$). Furthermore, if $f^{(n)}(z)$ is the n^{th} derivative of f at z_0 then this derivative can be computed from the series term by term. ["term by term differentiation"]. For example,

$$f'(z) = \sum_{\cancel{n \geq 0}\ n \geq 1} n a_n (z-z_0)^{n-1},$$

$$f''(z) = \sum_{\cancel{n \geq 0}\ n \geq 2} n(n-1) a_n (z-z_0)^{n-2},$$

and, the radius of convergence of the

derivative Taylor series' is the same as the radius of convergence of the first series!

Idea of proof:

$$f'(z) = \lim_{h\to 0} \frac{f(z+h) - f(z)}{h}$$

$$= \lim_{h\to 0} \frac{\sum_{n\geq 0} a_n (z+h-z_0)^n - \sum_{n\geq 0} a_n (z-z_0)^n}{h}$$

$$= \lim_{h\to 0} \sum_{n\geq 0} \frac{\left(a_n (z+h-z_0)^n - a_n (z-z_0)^n\right)}{h}$$

(True if both series converge absolutely)

$$= \lim_{h\to 0} \sum_{n\geq 0} \left[\frac{a_n(z+h-z_0)^n - a_n(z-z_0)^n}{h}\right]$$

we can switch these too

if:

$$\lim_{h\to 0} \sum_{n\geq 0} f_n(h) = \sum_{n\geq 0} \lim_{h\to 0} (f_n(h))$$

in various situations.

(eg: ① $f_n(h) \geq 0$.

or ② $\sum_{n\geq 0} |f_n(h)|$ converges uniformly on the set $|h| \leq \varepsilon$.)

$$= \sum_{n\geq 0} \lim_{h\to 0} \frac{a_n(z+h-z_0)^n - a_n(z-z_0)^n}{h}$$

$$= \sum_{n\geq 0} n\, a_n (z-z_0)^{n-1}. \quad \boxed{\text{QED}}$$

In our case, if we restrict to $\underbrace{|z-z_0|}_{} < r < R$.

Sums & sums of derivs converges absolutely & uniformly here.

(use comparison with geometric series —
— use integral formula for the coefficients.)

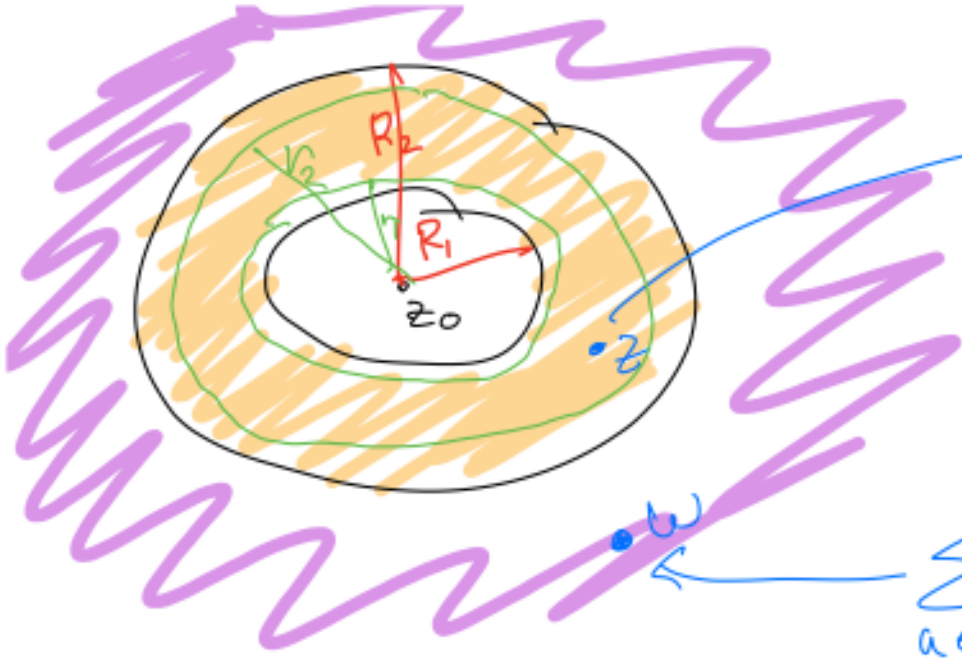
Laurent series version:

Given f holomorphic on an annulus $R_1 < |z - z_0| < R_2$,

we have $f(z) = \sum_{n \in \mathbb{Z}} a_n (z - z_0)^n$

on this annulus,

R_1 & R_2 can be chosen so that R_1 is minimal & R_2 is maximal for this series — Then f cannot be extended to be defined on a larger annulus.



$f(z) = \sum_{n\in\mathbb{Z}} a_n (z-z_0)^n$ ← converges here

$\sum_{n\in\mathbb{Z}} a_n (w-z_0)^n$ diverges.

If $R_1 < r_1 < |z-z_0| < r_2 < R_2$,

Then $\sum_{n\in\mathbb{Z}} a_n (z-z_0)^n$ converges absolutely & uniformly on

$$\{z : r_1 < |z-z_0| < r_2\}$$

an even on $\{z : r_1 \le |z-z_0| \le r_2\}$.

But it does not converge uniformly on the large annulus (unless it's a special case where the series is finite).

Similar to the Taylor series case, you can differentiate the Laurent series term by term, and the differentiated series converges on the same open annulus of convergence.

On the boundaries of these domains of convergence, the Taylor or Laurent series may converge or diverge (depending on the function and the point.) — The convergence properties on the boundary might change when you take derivatives.

$\boxed{\text{Example}}$ $f(z) = \sum_{n\geq 1} \frac{z^n}{n^2}$.

Radius of conv. $= 1$

converges on $B(0,1)$, diverges on $\mathbb{C} \setminus \overline{B(0,1)}$.

f actually converges absolutely & uniformly at every point of the circle $\partial B(0,1)$!

$$\left|\frac{z^n}{n^2}\right| = \frac{1}{n^2}, \quad \sum \frac{1}{n^2} < \infty.$$

Next, consider $f'(z) = \sum_{n\geq 1} \frac{n z^{n-1}}{n^2} = \sum_{n\geq 1} \frac{z^{n-1}}{n}$.

On the circle, $f'(z)$'s series diverges at $z=1$, $\sum \frac{1}{n} = \infty$.

But it converges at some other pts, such as $z=-1$, $\sum_{n\geq 1} \frac{(-1)^{n-1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \dots$

(converges by alt. series test.)

<u>Ex Credit:</u> Find all z (with proof) s.t. $\sum_{n\geq 1} \frac{z^{n-1}}{n}$ converges & $|z|=1$.

Next, consider $f''(z) = \sum_{n\geq 2} \frac{(n-1)z^{n-2}}{n}$,

This diverges at every z s.t. $|z|=1$.

Why? $\lim_{n\to\infty} \left|\frac{(n-1)z^{n-2}}{n}\right| = \lim_{n\to\infty} \frac{n-1}{n} = 1 \neq 0$

$\therefore \lim_{n\to\infty} \frac{(n-1)z^{n-2}}{n} \neq 0$.

$\therefore$ the series diverges.

$\boxed{\text{Example}}$ Find the Taylor series of

$g(z) = \frac{2}{(3-4z)^2}$ centered at the origin

$$\left[\frac{2}{(3-4z)}\right]' = \left[2(3-4z)^{-1}\right]'$$
$$= 2\cdot(3-4z)^{-2}\cdot 4(-1)$$

$$= \frac{8}{(3-4z)^2}$$

$$\therefore\ g(z) = \frac{2}{(3-4z)^2} = \frac{1}{4}\left[\frac{2}{(3-4z)}\right]'$$

$$= \frac{1}{4}\left[\frac{2}{3}\cdot\frac{1}{1-\left(\frac{4z}{3}\right)}\right]'$$

$$= \left[\frac{1}{6}\cdot\sum_{n\geq 0}\left(\frac{4z}{3}\right)^n\right]'$$

Need $\left|\frac{4z}{3}\right|<1$

$|z|<\frac{3}{4}$

$$= \frac{1}{\cancel{6}\,3}\sum_{\substack{\cancel{n\geq 0}\\ n\geq 1}} n\left(\frac{4z}{3}\right)^{n-1}\cdot\frac{\cancel{4}\,2}{3}$$

$$= \sum_{n\geq 1}\frac{2}{9}\,\frac{4^{n-1}}{3^{n-1}}\cdot n\cdot z^{n-1}$$

$k=n-1 \Rightarrow n=k+1$

$$g(z) = \boxed{\sum_{k\geq 0}\frac{2}{9}\,\frac{4^k}{3^k}(k+1)\,z^k}\ ;$$

which converges on $B(0, \frac{3}{4})$.